Exercise V
Contact mechanics

Question 1

The formulation of the shear stress in the Hertzian elastic contact model is

$$\tau = \frac{1}{2}\left|\sigma_z - \sigma_r\right| \tag{1}$$

where the normal stresses are

$$\sigma_r = p_0\left[(1+\nu)\left(\frac{z}{a}\tan^{-1}\left(\frac{a}{z}\right)-1\right)+\frac{1}{2}\frac{a^2}{a^2+z^2}\right]$$

$$\sigma_z = -p_0\frac{a^2}{a^2+z^2}$$

In the origin point $z=0$ and the previous expressions are

$$\sigma_r\left(z=0\right) = p_0\left[(1+\nu)\left(-1\right)+\frac{1}{2}\right] = -p_0\frac{1+2\nu}{2}$$

$$\sigma_z\left(z=0\right) = -p_0$$

Then the formulation (1) of the shear stress (from Mohr's circle) becomes

$$\tau\left(z=0\right) = \frac{1}{2}\left|-p_0 + p_0\frac{1+2\nu}{2}\right| = p_0\frac{1-2\nu}{4}$$

which for $\nu = 0.3$ is $\tau = 0.1p_0$.

Question 2

The complete expression of the shear stress is

$$\tau = \frac{1}{2}\left| -p_0 \frac{a^2}{a^2+z^2} - p_0 \left[(1+\nu)\left(\frac{z}{a}\tan^{-1}\left(\frac{a}{z}\right) - 1 \right) + \frac{1}{2}\frac{a^2}{a^2+z^2} \right] \right| \tag{2}$$

Because of the Hertzian model (circular contact) $\sigma_r = \sigma_\theta$ and then the maximum value is along the z-axis.
Calling $\xi = \frac{z}{a}$ it is possible to rewrite (2) in this form:

$$\begin{aligned} \tau &= \frac{1}{2}p_0 \left| \frac{1}{1+\xi^2} + \left[(1+\nu)\left(\xi \tan^{-1}\frac{1}{\xi} - 1 \right) + \frac{1}{2}\frac{1}{1+\xi^2} \right] \right| = \\ &= \frac{1}{2}p_0 \left| (1+\nu)\left(\xi \tan^{-1}\frac{1}{\xi} - 1 \right) + \frac{3}{2}\frac{1}{1+\xi^2} \right| \end{aligned}$$

Using the value $\nu = 0.3$ for the Poisson's ratio, the stress divided by the load is:

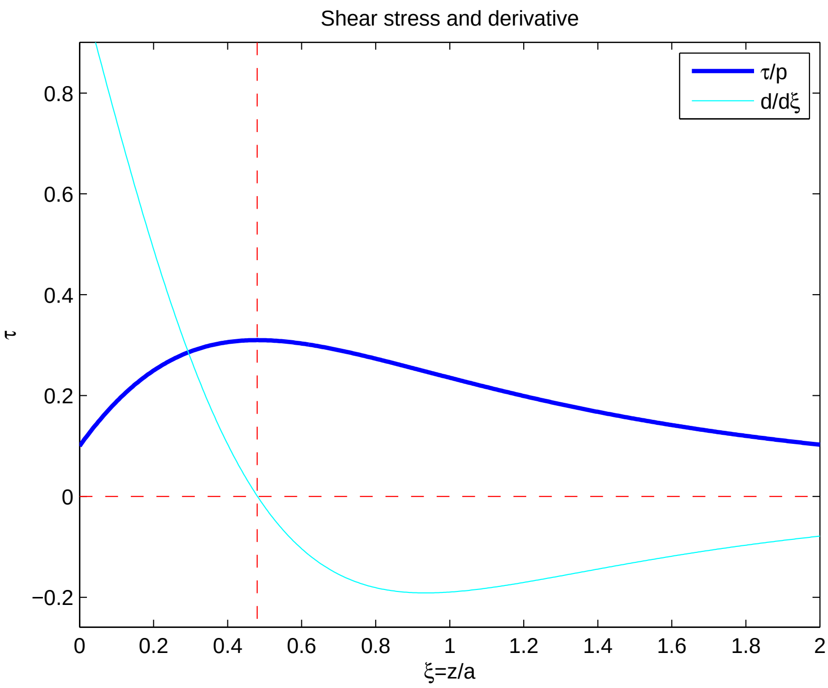
$$\frac{\tau}{p_0} = \frac{1}{2}\left| 1.3\left(\xi \tan^{-1}\frac{1}{\xi} - 1 \right) + \frac{3}{2}\frac{1}{1+\xi^2} \right| \tag{3}$$

To find the maximum value, the derivative can be calculated (it is sufficient to do this for $\xi > 0$, because the function is even) :

$$\frac{d}{d\xi}\left(\frac{\tau}{p_0}\right) = 0$$

$$0.65 \arctan\left(\frac{1}{\xi}\right) - \frac{3\,\xi}{2\,(\xi^2+1)^2} - \frac{0.65}{\xi\,\left(\frac{1}{\xi^2}+1\right)} = 0 \tag{4}$$

In the following figure the curves of (3) and (4) are shown:



Solving the equation (4), we can find the coordinate for the maximum shear stress:

$$\xi_0 = 0.4809$$

$$z_0 = 0.4809\,a$$

Note that the positive direction of the z-axis is toward the inside of the surface. In this point the value of τ is:

$$\tau \approx 0.31002\,p_0$$